NAG Toolbox for MATLAB

f04fe

1 Purpose

f04fe solves the Yule–Walker equations for a real symmetric positive-definite Toeplitz system.

2 Syntax

```
[x, p, v, vlast, ifail] = f04fe(n, t, wantp, wantv)
```

3 Description

f04fe solves the equations

$$Tx = -t,$$

where T is the n by n symmetric positive-definite Toeplitz matrix

$$T = \begin{pmatrix} \tau_0 & \tau_1 & \tau_2 & \dots & \tau_{n-1} \\ \tau_1 & \tau_0 & \tau_1 & \dots & \tau_{n-2} \\ \tau_2 & \tau_1 & \tau_0 & \dots & \tau_{n-3} \\ . & . & . & & . \\ \tau_{n-1} & \tau_{n-2} & \tau_{n-3} & \dots & \tau_0 \end{pmatrix}$$

and t is the vector

$$t^{\mathrm{T}} = (\tau_1, \tau_2 \dots \tau_n).$$

The function uses the method of Durbin (see Durbin 1960 and Golub and Van Loan 1996). Optionally the mean square prediction errors and/or the partial correlation coefficients for each step can be returned.

4 References

Bunch J R 1985 Stability of methods for solving Toeplitz systems of equations *SIAM J. Sci. Statist. Comput.* **6** 349–364

Bunch J R 1987 The weak and strong stability of algorithms in numerical linear algebra *Linear Algebra Appl.* **88/89** 49–66

Cybenko G 1980 The numerical stability of the Levinson–Durbin algorithm for Toeplitz systems of equations *SIAM J. Sci. Statist. Comput.* **1** 303–319

Durbin J 1960 The fitting of time series models *Rev. Inst. Internat. Stat.* **28** 233

Golub G H and Van Loan C F 1996 *Matrix Computations* (3rd Edition) Johns Hopkins University Press, Baltimore

5 Parameters

5.1 Compulsory Input Parameters

1: **n – int32 scalar**

The order of the Toeplitz matrix T.

Constraint: $\mathbf{n} \geq 0$. When $\mathbf{n} = 0$, then an immediate return is effected.

2: **t(0 : n) – double array**

t(0) must contain the value τ_0 of the diagonal elements of T, and the remaining **n** elements of **t** must contain the elements of the vector t.

Constraint: $\mathbf{t}(0) > 0.0$. Note that if this is not true, then the Toeplitz matrix cannot be positive-definite.

3: **wantp – logical scalar**

Must be set to **true** if the partial (auto)correlation coefficients are required, and must be set to **false** otherwise.

4: **wantv – logical scalar**

Must be set to **true** if the mean square prediction errors are required, and must be set to **false** otherwise.

5.2 Optional Input Parameters

None.

5.3 Input Parameters Omitted from the MATLAB Interface

work

5.4 Output Parameters

1: **x(*) – double array**

Note: the dimension of the array **x** must be at least $\max(1,\mathbf{n})$.

The solution vector x.

2: **p(*) – double array**

Note: the dimension of the array **p** must be at least $\max(1,\mathbf{n})$ if **wantp** = **true**, and at least 1 otherwise.

With **wantp** as **true**, the ith element of **p** contains the partial (auto)correlation coefficient, or reflection coefficient, p_i for the ith step. (See Section 8 and Chapter G13.) If **wantp** is **false**, then **p** is not referenced. Note that in any case, $x_n = p_n$.

3: **v(*) – double array**

Note: the dimension of the array **v** must be at least $\max(1,\mathbf{n})$ if **wantv** = **true**, and at least 1 otherwise.

With **wantv** as **true**, the ith element of **v** contains the mean square prediction error, or predictor error variance ratio, v_i, for the ith step. (See Section 8 and Chapter G13.) If **wantv** is **false**, then **v** is not referenced.

4: **vlast – double scalar**

The value of v_n, the mean square prediction error for the final step.

5: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Note: f04fe may return useful information for one or more of the following detected errors or warnings.

ifail $= -1$

On entry, **n** < 0,
or **t**$(0) \leq 0.0$.

ifail > 0

The principal minor of order (**ifail** $+ 1$) of the Toeplitz matrix is not positive-definite to working accuracy. If, on exit, $x_{\mathbf{ifail}}$ is close to unity, then the principal minor was close to being singular, and the sequence $\tau_0, \tau_1, \ldots, \tau_{\mathbf{ifail}}$ may be a valid sequence nevertheless. The first **ifail** elements of **x** return the solution of the equations

$$T_{\mathbf{ifail}} x = -\left(\tau_1, \tau_2, \ldots, \tau_{\mathbf{ifail}}\right)^{\mathrm{T}},$$

where $T_{\mathbf{ifail}}$ is the **ifail**th principal minor of T. Similarly, if **wantp** and/or **wantv** are true, then **p** and/or **v** return the first **ifail** elements of **p** and **v** respectively and **vlast** returns $v_{\mathbf{ifail}}$. In particular if **ifail** $=$ **n**, then the solution of the equations $Tx = -t$ is returned in **x**, but $\tau_{\mathbf{n}}$ is such that $T_{\mathbf{n}+1}$ would not be positive-definite to working accuracy.

7 Accuracy

The computed solution of the equations certainly satisfies

$$r = Tx + t,$$

where $\|r\|_1$ is approximately bounded by

$$\|r\|_1 \leq c\epsilon\left(\prod_{i=1}^{n}(1 + |p_i|) - 1\right),$$

c being a modest function of n and ϵ being the ***machine precision***. This bound is almost certainly pessimistic, but it has not yet been established whether or not the method of Durbin is backward stable. If $|p_n|$ is close to one, then the Toeplitz matrix is probably ill-conditioned and hence only just positive-definite. For further information on stability issues see Bunch 1985, Bunch 1987, Cybenko 1980 and Golub and Van Loan 1996. The following bounds on $\|\mathbf{t}^{-1}\|_1$ hold:

$$\max\left(\frac{1}{v_{n-1}}, \frac{1}{\prod_{i=1}^{n-1}(1 - p_i)}\right) \leq \|T^{-1}\|_1 \leq \prod_{i=1}^{n-1}\left(\frac{1 + |p_i|}{1 - |p_i|}\right).$$

Note: $v_n < v_{n-1}$. The norm of T^{-1} may also be estimated using function f04yc.

8 Further Comments

The number of floating-point operations used by f04fe is approximately $2n^2$, independent of the values of **wantp** and **wantv**.

The mean square prediction error, v_i, is defined as

$$v_i = (\tau_0 + (\tau_1\tau_2 \ldots \tau_{i-1})y_{i-1})/\tau_0,$$

where y_i is the solution of the equations

$$T_i y_i = -(\tau_1\tau_2 \ldots \tau_i)^{\mathrm{T}}$$

and the partial correlation coefficient, p_i, is defined as the ith element of y_i. Note that $v_i = \left(1 - p_i^2\right)v_{i-1}$.

9 Example

```
n = int32(4);
t = [4;
     3;
     2;
     1;
     0];
wantp = true;
wantv = true;
[x, p, v, vlast, ifail] = f04fe(n, t, wantp, wantv)
```

```
x =
   -0.8000
    0.0000
   -0.0000
    0.2000
p =
   -0.7500
    0.1429
    0.1667
    0.2000
v =
    0.4375
    0.4286
    0.4167
    0.4000
vlast =
    0.4000
ifail =
           0
```